CONCERNING EQUILIBRIUM CRACKS FORMING DURING BRITTLE FRACTURE. THE STABILITY OF ISOLATED CRACKS. RELATIONSHIPS WITH ENERGETIC THEORIES

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1\. 1. Let a rectilinear crack in an infinite flat plate by supported in an open state by breaking loads, symmetrical relative to a straight line along which the crack lies and relative to the center of the crack. As was shown earlier [1] the half-length of such a crack l is determined by the relationship*

$$\int_0^l \frac{p(x)\,dx}{\sqrt{l^2 - x^2}} = \frac{K}{\sqrt{2l}} \tag{1.1}$$

where K is the modulus of cohesion [1,2], the coordinate x is the distance along the crack from the center, $p(x)$ is the distribution of the normal stresses appearing along the x-axis in the continuous plate without a crack under the action of the same stresses. The function $p(x)$ is easily found for the given loads and can therefore be considered known.

Let us assume that the acting loads are proportional to a certain parameter λ; it is evident that $p(x)$ will also be proportional to λ, so that $p(x) = \lambda f(x)$.

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* Note that the whole consideration in reference [1] was carried out for the case of plane deformation (thick plate). For the conversion of results to the case of the generalized plane state of stress (thin plate), it is sufficient to substitute $E/(1-\nu)$ for E and K for $K_1 = K\sqrt{1-\nu^2}$. In particular, the formula for the length of the crack in the problem on the cracking of the plate in the case of the generalized plane stress takes the form $L = E^2 h^2 / 4K^2$.

Converting to the dimensionless variable $\xi = x/l$, we reduce equation (1.1) to the form

$$\Phi(l) = \sqrt{l}\int_0^1 \frac{f(l\xi)\,d\xi}{\sqrt{1-\xi^2}} = \frac{K}{\sqrt{2}\lambda} \tag{1.2}$$

Further, the radius R of a circular crack in an infinite body, supported in an open state by an axially-symmetric breaking load, which is also symmetric relative to the plane of the crack, is determined by the relationship [2]

$$\int_0^R \frac{rp(r)\,dr}{\sqrt{R^2-r^2}} = K\sqrt{\frac{R}{2}} \tag{1.3}$$

where $p(r)$ is the distribution of normal stresses in the plane of symmetry of the load for the continuous body without a crack under the action of the same load. If again the acting load is proportional to the parameter λ, then the function $p(r)$ is also proportional to the parameter λ, so that $p(r) = \lambda f(r)$ and equation (1.3) is reduced to the form

$$\Phi(R) = \sqrt{R}\int_0^1 \frac{\xi f(R\xi)\,d\xi}{\sqrt{1-\xi^2}} = \frac{K}{\sqrt{2}\lambda} \tag{1.4}$$

Therefore, in all cases the relationship determining the crack dimension has the form:

$$\Phi(c) = K/\sqrt{2}\lambda \tag{1.5}$$

where under c one must understand the half-length of a rectilinear crack l or the radius of a circular crack R, and the function $\Phi(c)$ for the rectilinear and ciruclar cracks is determined, respectively, by the equations

$$\Phi(c) = \sqrt{c}\int_0^1 \frac{f(c\xi)\,d\xi}{\sqrt{1-\xi^2}}, \qquad \Phi(c) = \sqrt{c}\int_0^1 \frac{\xi f(c\xi)\,d\xi}{\sqrt{1-\xi^2}} \tag{1.6}$$

2. The study of the dependence of the crack dimension upon the load leads to the study of the functions $\Phi(c)$ determined by the equations (1.6).

Let us exclude from consideration the case when the crack is formed by concentrated forces applied on its surface; these cases are considered in sufficient detail in references [1,2]. Let the crack, therefore, be maintained in an open state by some forces, in particular it may be by concentrated forces applied inside the body, and by distributed loads applied at the surface of the crack. Moreover, the function p and consequently also the function f will be limited.

At small values of c, we obtain respectively in the first and the second cases from equations (1.6)

$$\Phi(c) \approx \sqrt{c}\, f(0) \int_0^1 \frac{d\xi}{\sqrt{1-\xi^2}} = \frac{\pi}{2} f(0)\sqrt{c}, \qquad \Phi(c) \approx f(0)\sqrt{c} \qquad (1.7)$$

so that at small c the functions $\Phi(c)$ grow proportionally to $\sqrt{c}$. If the breaking forces applied to the body from each side of the crack are limited and for definiteness are equal to λP, then the following relationship holds

$$\int_{-\infty}^{\infty} p(x)\,dx = \lambda P, \qquad \int_0^{\infty} f(c\xi)\,d\xi = \frac{P}{2c} \qquad (1.8)$$

$$\int_0^{\infty} p(r)\,r\,dr = \frac{\lambda P}{2\pi}, \qquad \int_0^{\infty} f(c\xi)\,\xi\,d\xi = \frac{P}{2\pi c^2}$$

Hence from (1.6) we also obtain respectively in the first and second cases as $c \to \infty$

$$\Phi(c) \sim \frac{P}{2\sqrt{c}}, \qquad \Phi(c) \sim \frac{P}{2\pi c^{3/2}} \qquad (1.9)$$

so that in both cases the functions $\Phi(c)$ at infinity decrease, converging to zero, and have at least one positive finite maximum. We assume that the largest of the maxima is reached at some point $c = c_0$ (c_0 known to be positive), at which the value of the function $\Phi(c)$ is equal to Φ_0. Then at $\lambda < \lambda_0 = K/\sqrt{2}\,\Phi_0$ equation (1.5) does not have a solution; this means that at too small loads an equilibrium crack generally does not form. At $\lambda = \lambda_0$ there is one root $c = c_0$ of this equation. Physically this means that at reaching a certain critical load a crack of a definite finite dimension is now formed immediately and abruptly.

For $\lambda > \lambda_0$, equation (1.5) has several roots. Which of these roots really correspond to the actual crack is determined by a consideration of stability.

In the case when the applied forces, acting on the body from both sides of the crack, are not limited, the function $\Phi(c)$ cannot have decreasing portions. It will be thus, in particular, in the case of a homogeneous field when $\Phi(c)$ is proportional to $\sqrt{c}$.

2. The equilibrium state of a crack according to definition is stable if any sufficiently small change in dimension of crack leads the appearances of forces striving to return the system to the state of equilibrium that has been disturbed. We shall investigate the condition of stability for cases of a rectilinear crack in an infinite flat plate and a circular crack in an infinite body for arbitrary symmetrical breaking loads.

Let the loads again be proportional to some parameter λ, so that an increase in the parameter λ corresponds to an increase in load.

For the stability of a crack it is necessary that the equilibrium dimension of the crack grow with an increase of the parameter λ. Actually, we assume that with an increase of load the corresponding equilibrium dimension c increases. Let the dimension of the crack be slightly decreased in comparison with the equilibrium dimension at the same load. Moreover, the total force connecting both halves of the body together is decreased, the equilibrium with the applied, somewhat larger, loads is disturbed and the crack will tend to spread. If the crack dimension is somewhat increased in comparison with the equilibrium dimension, then the equilibrium is disturbed in the reverse direction and the crack will tend to close up. If the equilibrium dimension of the crack is decreased with an increase in load, when the situation is close to a given equilibrium state, then, evidently, at a small change in crack dimension the forces produced will aggravate the deviation from equilibrium condition and the crack will abruptly or catastrophically spread so that the corresponding equilibrium state will be unstable.

Thus, the equilibrium state, corresponding to some dimension of crack c and to the corresponding value of the parameter λ, is stable if for the given c and λ the following condition is fulfilled

$$\frac{dc}{d\lambda} > 0 \tag{2.1}$$

Differentiating equation (1.5), we obtain

$$\Phi'(c)\frac{dc}{d\lambda} = -\frac{K}{\sqrt{2}\,\lambda^2} \qquad \text{or} \qquad \frac{dc}{d\lambda} = -\frac{K}{\sqrt{2}\lambda^2\Phi'(c)} \tag{2.2}$$

Hence from the inequality (2.1) we also obtain the condition of crack stability in the form

$$\Phi'(c) < 0 \tag{2.3}$$

Therefore, only those equilibrium states are stable which correspond to the decreasing portion of the curve $\Phi(c)$. Hence, in particular, it follows that if the crack is maintained in an open state by forces applied inside the body, and propagated by loads applied on the surface of the crack, and if the forces applied from each side of the crack are limited, then at loads greater than the critical there is at least one stable and one unstable equilibrium state. In case of a homogeneous field all equilibrium states are unstable because in this case the function $\Phi(c)$ is proportional to $\sqrt{c}$ and its derivative is positive for all values of c.

The critical equilibrium states, separating the stable from the unstable states are determined by the relationship (1.5) and also by the requirement of the maxima and minima of the function $\Phi(c)$ for these states.

In the transition through the critical equilibrium states, the crack dimensions change abruptly, - the point representing the equilibrium state passes from one stable part of the curve $\Phi(c)$ to another. Therefore, the graph of the function $\Phi(c)$ makes possible a complete representation of the picture of crack development with increasing load. Generally speaking, the number of extrema of the function $\Phi(c)$ may be arbitrarily large; therefore, this picture may appear quite complex. If the crack is reversible, then using the graph of the function $\Phi(c)$, one may also study the picture of the change in crack dimensions for a decrease and any non-monotonic change in load. Moreover, it is evident that the abrupt change in the crack dimensions will occur, generally speaking, not for a monotonic increase in load, but rather in other cases. The resulting conditions of stability make it possible also to judge the crack stability in finite bodies, because the effect of boundaries can replace the action of corresponding forces in an infinite body.

3. In almost all investigations without exception which have shed light on the formation and development of cracks, beginning with the classic work of Griffith [3], the energetic approach was used, the idea of which is essentially the following. Let W be the decrease in elastic energy of the body due to the formation of the crack, and U the surface energy of the crack; at a given crack configuration the values of W and U will depend only on the crack dimension c. For the equilibrium state the maximum and minimum condition of free energy must be fulfilled

$$\frac{\partial}{\partial c}(W-U)=0 \tag{3.1}$$

which also determines the relationship of load and crack dimension.

Under the assumption that the supplementary stresses caused by the cohesion forces add an essential contribution to the increase of elastic energy because of the crack formation and that the density of the surface energy is constant, the result is that W does not depend upon the cohesion force and is determined by the solution of a problem in elasticity theory for a given load and crack configuration, and $U = 2TS$, where S is the area of the crack in a plane. Considering the rupture to be ideally brittle, Griffith [3] identified the density of the surface energy T with the surface tension of the material. Orowan [4] and Irwin [5] extended Griffith's conception to incompletely brittle materials, considering $2T$ to be the specific work of plastic deformations in a thin layer close to the crack surface.

The energetic approach to the solution of the problems in the development of cracks is essentially more complex than the proposed force approach, because of the necessity of using the elastic energy. This, in particular, is evident from the fact that hitherto in fact the determination of the crack dimensions by the energetic method has been done only for the trivial and, moreover, unstable case of a homogeneous field.

We shall show that the force approach does not contradict the energetic approach. Indeed, from the smallness of the terminal region (first hypothesis [2]) and final stress in this region (third hypothesis) follows a small contribution to the elastic energy, introduced by the presence of cohesion forces. From the autonomous final region of the crack (second hypothesis) follows the constancy of the work T, done against the cohesion forces in the creation of a unit surface of crack, so that the work spent on surmounting the cohesion forces during the creation of a crack with an area in the plane S, - and this also is the surface energy of the crack in conformity with its definition, - is equal to $2TS$.

Therefore, for the establishment of the relationship of the modulus of cohesion K with the density of the surface energy T one must, evidently, compare the relationships determining the crack dimensions obtained by the energetic and the force methods for any problem, for example, for the problem concerning the ciruclar crack in an infinite body maintained by an arbitrary axially-symmetric rupturing load which is symmetric with respect to the plane of the crack.

We shall represent the state of stress in a body with a crack in the form of the sum of two states of stress: the state of stress in a continuous body with a given load and the state of stress in the body with a crack on the surface of which normal stresses $p(r)$ are applied.

What is interesting is not the increase itself of the elastic energy of the total state of stress, but the derivative of this increase with respect to the dimension, in the given case with respect to the crack radius (rate of release of the elastic energy); therefore the calculation of the increase of elastic energy of the total state of stress can replace the calculation of the increase of elastic energy of any state of stress differing from the total in magnitude not dependent upon the crack radius. The first state of stress, obviously does not depend on the crack radius, hence one may take as such a state of stress the second stress state. In general form this convenient method was first proposed by Bueckner [6].

Let us calculate the elastic energy of the second state of stress. If on the surface of the crack of radius R normal stresses $-\epsilon p(r)$ are applied, then as shown by Sneddon [7], the normal displacements of points of the crack surface are equal to

$$w = \frac{4(1-\nu^2)R\varepsilon}{\pi E}\int_\rho^1 \frac{\mu d\mu}{\sqrt{\mu^2-\rho^2}}\int_0^1 \frac{\xi p(\xi\mu R)\, d\xi}{\sqrt{1-\xi^2}}, \qquad \rho = \frac{r}{R} \tag{3.2}$$

and the changes in the normal displacements, corresponding to the increase in load up to $-(\epsilon + d\epsilon)p(r)$, are equal to

$$dw = \frac{4(1-\nu^2)R\, d\varepsilon}{\pi E}\int_\rho^1 \frac{\mu d\mu}{\sqrt{\mu^2-\rho^2}}\int_0^1 \frac{\xi p(\xi\mu R)\, d\xi}{\sqrt{1-\xi^2}} \tag{3.3}$$

It is easy to see that the decrease in elastic energy caused by the formation of the crack is equal to

$$W = 2\int_0^1 \varepsilon d\varepsilon \int_0^R 2\pi r p(r)\, \frac{4(1-\nu^2)R}{\pi E}\, dr \int_\rho^1 \frac{\mu d\mu}{\sqrt{\mu^2-\rho^2}} \times$$
$$\times \int_0^1 \frac{\xi p(\xi\mu R)\, d\xi}{\sqrt{1-\xi^2}} = \frac{8(1-\nu^2)R^3}{E}\int_0^1 p(\rho R)\rho d\rho \int_\rho^1 \frac{\mu d\mu}{\sqrt{\mu^2-\rho^2}}\int_0^1 \frac{\xi p(\xi\mu R)\, d\xi}{\sqrt{1-\xi^2}}$$

Changing the order of integration, we find

$$W = \frac{8(1-\nu^2)R^3}{E}\int_0^1 \mu d\mu \int_0^1 \frac{\xi\, p(\xi\mu R)\, d\xi}{\sqrt{1-\xi^2}}\int_0^\mu \frac{p(\rho R)\rho d\rho}{\sqrt{\mu^2-\rho^2}} \tag{3.4}$$

Setting $\rho = \mu\xi$, we have

$$\int_0^\mu \frac{p(\rho R)\rho d\rho}{\sqrt{\mu^2-\rho^2}} = \mu\int_0^1 \frac{\xi p(\xi\mu R)\, d\xi}{\sqrt{1-\xi^2}} = \mu F(\mu R)$$

whence we also obtained from (3.4)

$$W = \frac{8(1-\nu^2)R^3}{E}\int_0^1 \mu^2 F^2(\mu R)\, d\mu \tag{3.5}$$

Differentiating (3.5) and then integrating by parts, we have

$$\frac{\partial W}{\partial R} = \frac{8(1-\nu^2)}{E}\, 3R^2 \int_0^1 \mu^2 F^2(\mu R)\, d\mu +$$
$$+ \frac{8(1-\nu^2)R^3}{E}\int_0^1 2\mu^3 F F'\, d\mu = \frac{8(1-\nu^2)}{E} R^2 F^2(R) \tag{3.6}$$

Further we have

$$U = 2\pi R^2 T, \qquad dU/dR = 4\pi R T \tag{3.7}$$

whence also from the equilibrium conditions

$$\frac{\partial}{\partial R}(W - U) = 0$$

we obtain

$$F^2(R) = \frac{\pi ET}{2(1-\nu^2)R};$$

or

$$\left(\int_0^1 \frac{\xi p(\xi\mu R)\, d\xi}{\sqrt{1-\xi^2}}\right)^2 = \frac{1}{2R}\frac{\pi ET}{1-\nu^2} \tag{3.8}$$

Finally we find, passing to the variable $r = \xi R$:

$$\int_0^R \frac{rp(r)\, dr}{\sqrt{R^2 - r^2}} = \sqrt{\frac{R}{2}}\sqrt{\frac{\pi ET}{1-\nu^2}} \tag{3.9}$$

Since this relationship must coincide with the equation (1.3), the modulus of cohesion K must be related to the density of surface energy T, Young's modulus E and Poisson's ratio ν by the equation

$$K^2 = \frac{\pi ET}{1-\nu^2} \tag{3.10}$$

Since this relationship relates the universal characteristics of a medium, it also must be universal.

4. Completely analogously, though technically somewhat more complicated one may obtain by the energetic method equation (1.1), determining the dimension of an isolated rectilinear crack in the case of plane deformation for an arbitrary distribution of rupturing stresses.

The equation for the change of the elastic energy is obtained in this case in the form

$$W = \frac{2(1-\nu^2)l^2}{\pi E}\int_0^\pi p(l\cos\theta)\sin\theta d\theta \int_0^\pi p(l\cos\varphi)\sin\varphi \ln\frac{\sin {}^1/_2(\varphi+\theta)}{\sin {}^1/_2(\varphi-\theta)}d\varphi + \ldots \tag{4.1}$$

where $x = l\cos\theta$, and the dotted line indicates terms not dependent on the crack dimension. Differentiating with respect to l and integrating by parts thereafter, we obtain an equation for the rate of release of elastic energy

$$\frac{\partial W}{\partial l} = \frac{8(1-\nu^2)l}{\pi E}\left\{\int_0^l \frac{p(x)\, dx}{\sqrt{l^2-x^2}}\right\}^2 \tag{4.2}$$

Whence, using the equation of the surface energy $U = 4Tl$ and equilibrium conditions (3.1), we obtain the equation

$$\int_0^l \frac{p(x)\,dx}{\sqrt{l^2-x^2}} = \sqrt{\frac{\pi ET}{1-\nu^2}}\frac{1}{\sqrt{2l}} \tag{4.3}$$

which coincides with equation (1.1) if the cohesion modulus K is related to the density of surface energy and elastic characteristics of the material by the expression (3.10). The problem of determining the crack dimensions in an arbitrary field of rupturing stresses has been considered repeatedly; an attempt to solve it was proposed not long ago by Masubuchi [8]. Using the expansion of stresses and displacements of the points of the crack surface in a trigonometric series, Masubuchi obtained an equation for the rate of release of the elastic energy also in the form of a series whose terms were expressed by a coefficient of the mentioned trigonometric series. The inadequate effectiveness of the approach did not permit Masubuchi to obtain a simple resulting equation (1.1).

Therefore, we come to the following conclusion. The clarification of the physical picture close to the ends of a crack (1.2) permits one to consider the problem concerning equilibrium cracks as a problem in elasticity theory, adding to the characteristic properties of a material a new characteristic, the cohesion modulus K. Such a force approach does not contradict the energetic approach developed in the preceding work [3-5]; however, being essentially more effective, the force approach makes the energetic approach unsuitable.

One must note that repeating the energetic derivations of the equations given above, which relate the load with the crack dimensions, one can obtain corresponding conditions regarding the finiteness of the stress [1,2] from the maximum and minimum condition of the total elastic energy. The total elastic energy is governed by the loads and the cohesion forces, considering the cohesion forces acting in the end region; moreover, the cohesion forces are assumed to be surface forces and normal to the crack surface. On the other hand, from the condition of the finiteness of the stress at the edges of the crack one can obtain the maximum and minimum condition of the total elastic energy. This shows the equivalence of the force and energetic approaches.

The solution of concrete problems uncovers various possibilities of experimental determination of the cohesion modulus K; on the strength of equation (3.10) each such determination also gives the density of surface energy T. We notice that such a determination of surface energy density will be based on the rigorous solution of a problem in elasticity theory. At the present time there exists a single method of determination of surface energy proposed by Obreimow [9] and based on the approximate solution of the problem by the methods of strength of materials.

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BIBLIOGRAPHY

1. Barenblatt, G.I., O ravnovesnykh treshchinakh, obrazuiushchikhsia pri khrupkom razrushenii. Priamolineinye treshchiny v ploskikh plastinkakh (On equilibrium cracks formed in brittle fracture. Rectilinear cracks in flat plates). ***PMM*** Vol. 23, No. 4, 1959.

2. Barenblatt, G.I., O ravnovesnykh treshchinakh, obrazuiushchikhsia pri khrupkom razrushenii. Obshchie predstavleniia and gipotezy. Osesimmetrichnye treshchiny (On equilibrium cracks formed in brittle fracture. General assumptions and theories).Axially-symmetrical cracks). ***PMM*** Vol. 23, No. 3, 1959.

3. Griffith, A.A., the phenomenon of rupture and flow in solids. *Phil. Trans. Roy. Soc., Ser. A.* Vol. 221, 1920.

4. Orowan, E., Energy criteria of fracture. *Welding Journal*, Research Supplement, March, 1955.

5. Irwin, G.R., Analysis of stresses and strains near the end of a crack traversing a plate. *J. Appl. Mech.* Vol. 24, No. 3, 1957.

6. Bueckner, H.F., The propagation of cracks and the energy of elastic deformation. *Trans. ASME* Vol. 80, No. 6, 1958.

7. Sneddon, I., *Fourier Transforms*, N.Y., 1951.

8. Masubuchi, Koichi, Dislocation and strain energy release during crack propagation in residual stress field. *Proc. 8th Japan Nat. Congress for Appl. Mech.* 1958.

9. Obreimow, J., On the splitting strength of mica. *Proc. Roy. Soc. Ser. A.* Vol. 127, 1930.

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